

Technical Appendix to:
Should the Social Security Trust Fund hold Equities?
An Intergenerational Welfare Analysis

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A1. The Log-Linear Approximation

The log-linearized model can be summarized as follows. First, wages and the return on aggregate capital are technologically determined by the capital labor ratio and the valuation risk from (6-7),

$$(A.1a) \quad (\hat{w}/\hat{A})_t = \hat{k}_t - \hat{a}_t,$$

$$(A.1b) \quad \hat{R}_{t+1}^k = -R^k \hat{k}_{t+1} + R^k \hat{a}_{t+1} + v/R^k \hat{v}_{t+1},$$

where $R^k = (1-v/R^k)(1-\delta) > 0$.

To satisfy the Euler equations (4), the bonds and equity returns must vary with the capital-labor ratio in the same way as the return on capital. Their dependence on shocks is different, however, and they generally have non-zero intercept terms because of risk premiums. Specifically, safe bonds have a predictable return

$$(A.1c) \quad \hat{R}_{t+1}^b = R_{b,0t} - R^k \hat{k}_{t+1}$$

and equity has a return that depends on leverage and on the relative-return shock,

$$(A.1d) \quad \hat{R}_{t+1}^e = R_{e,0t} - R^k \hat{k}_{t+1} + R^k \hat{a}_{t+1} + v/R^k \hat{v}_{t+1} + R_{e,\mu t} \hat{\mu}_{t+1}$$

where $R_{e,\mu t} = [(1-v/R^k) + v\mu v/R^k] > 0$.

Second, consider the decision problem of the young. In equilibrium, bond and equity prices must be such that the young hold the entire net government debt and all capital except for social security equity holdings. For given policy rules, the consumption-savings decision of the young can be interpreted as a decision about how wages minus cash flows to the government are divided between consumption and aggregate capital investment. Let

$$y_t^1 = w_t/A_t (1-\tau_t) - \tau_t/A_t - (d_t - b_t \tau_t) + e_t \tau_t$$

be the wage income minus total cash-flows to the government, all defined as productivity ratios. Then the decision problem of the young is to divide

their exogenous "disposable income" $y_t^1 = c_t^1/A_t + k_{t+1}$ into consumption and investment.¹ The deviations of y_t^1 from the steady state are given by

$$(A.1e) \quad \hat{y}_t^1 = y_{10} + y_{1k} \hat{k}_t + y_{1a} \hat{a}_t + y_{1v} \hat{v}_t + y_{1\mu} \hat{\mu}_t$$

where

$$y_{1a} = - \frac{1}{a} + (1-\alpha) \left\{ \frac{R^b}{an} (d-\alpha) + \frac{R^k}{an} e (1-v/R^k) \right\} / y^1,$$

$$y_{1k} = \frac{1}{k} + [(1-\alpha) (1-v/R^k) + \alpha] \frac{R^b}{an} (d-\alpha) / y^1,$$

$$y_{1v} = y_{1R} v/R^k, \text{ and } y_{1\mu} = y_{1R} R e, \mu,$$

where $an = (1+n)(1+a)$, and $y_{1R} = \frac{1}{R} e_{t-1} R^k / an / y^1$. Note that $y_{1R} > 0$ iff $e > 0$. Thus, y_{1a} is negative for sufficiently small d and α and rising with d and e ; y_{1v} and $y_{1\mu}$ are positive and proportional to e .

It is straightforward to show that the young divide their income between consumption and savings such that the coefficients are proportional to the y^1 -coefficients (see Bohn 1998). For all state variables $z = k, a, v, \mu$, the coefficients for \hat{k}_{t+1} are

$$(A.1f) \quad k_z = \frac{1 + k/(c^1/A)}{c_{2k} + Rk/ + k/(c^1/A)} y_{1z}$$

and the coefficients for the consumption-productivity ratio $(c^1/A)_t$ are

$$(A.1g) \quad c_{1Az} = \frac{(1+k/(c^1/A)) (c_{2k} + Rk/)}{c_{2k} + Rk/ + k/(c^1/A)} y_{1,zt}.$$

Relative to the y_{1z} coefficients, the capital coefficients are scaled up or down depending on whether $c_{2k} + Rk/$ is above or below one. If the elasticity of substitution $1/\alpha$ is high, the k_z -coefficients are small and the c_{1Az} -coefficients are large. Then the capital labor ratio quickly moves towards its steady state after any disturbance, and individuals are willing to tolerate the implied variations in consumption.

Third, consider the consumption of the old. Since period- t productivity is known when generation t makes savings and investment decisions, it is convenient to focus on the ratio of old-age consumption to

¹ I call y_t^1 the young generation's "disposable income," although the cash flows to the government include voluntary debt and equity transactions. Intuitively, one can think of individuals as doing all capital investment and selling some of their equity claims to social security.

lagged productivity, c_{t+1}^2/A_t , rather than the ratio to current productivity. Its deviation from the steady state, $\hat{c}_{t+1}^2 = (\hat{c}^2/A)_{t+1} + \hat{a}_{t+1}$, is given by

$$(A.1h) \quad \hat{c}_{t+1}^2 = c_{2,0t} + c_{2,kt} \hat{k}_t + c_{2,at} \hat{a}_t + c_{2,vt} \hat{v}_t + c_{2,\mu t} \hat{\mu}_t$$

where

$$c_{2a} = (1-\alpha) \left\{ 1 - \frac{R^k k v / \alpha n + R^k e (1-v/R^k) + R^b (d-)}{(c^2/A-)} \right\},$$

$$c_{2v} = v/R^k \frac{R^k (k- e)}{(c^2/A-)}, \quad c_{2\mu} = - \frac{R e, \mu R^k e}{(c^2/A-)},$$

$$c_{2k} = \alpha + (1-\alpha) \frac{v/\alpha n k}{(c^2/A-)} - [(1-\alpha)(1-v/R^k) + \alpha] \frac{R^b (d-)}{(c^2/A-)},$$

and $(c^2/A-)$ denotes the steady state of c_{t+1}^2/A_t .

These coefficients confirm the intuition explained above. If $e < k$ (the trust fund holds less than the entire capital stock), $v > 0$, and $d-$ is reasonably small, c_{2a} and c_{2v} are positive and declining in e . That is, the old are exposed to productivity and valuation risk and their exposure declines if the trust fund hold more equities, as claimed above. Their exposure to productivity risk also declines with $d-$, i.e., when there is more public debt. Finally, the μ_t -shocks are irrelevant if $e=0$ ($c_{2\mu}=0$); but for $e > 0$, $c_{2\mu}$ is negative. This negative exposure to relative return risk is an unavoidable side-effect of trust fund equity investment.

Since the old hold all private wealth, the risk premiums in asset returns depend on the conditional covariances between returns and old-age consumption. Assuming log-normality, one obtains

$$(A.1i) \quad PR^e_t \ln(E_t[R^e_{t+1}]) - \ln(R^b_{t+1}) = COV_t(\hat{R}^e_{t+1}, \hat{c}_{t+1}^2).$$

$$\text{and} \quad PR^k_t \ln(E_t[R^k_{t+1}]) - \ln(R^b_{t+1}) = COV_t(\hat{R}^k_{t+1}, \hat{c}_{t+1}^2)$$

Using (A.1b,d,h), these covariances can be evaluated as functions of the covariance matrix of shocks weighted by elasticity coefficients.

For the welfare analysis, it is useful to apply a common factor to young and old consumption. In analogy to \hat{c}_t^2 , let $\hat{c}_t^1 = (\hat{c}^1/A)_{t+1} + \hat{a}_t$, denote the log-deviations of c_t^1/A_{t-1} from its steady state. When \hat{a}_t is added, the

a-coefficient in the law of motion is raised by one while the other coefficients remain unchanged; i.e., \hat{c}_t^1 has coefficients $c_{1a} = c_{1Az}+1$ and $c_{1z} = c_{1Az}$ for $z=k,v,\mu$.² Overall, equations (A.1a-i) characterize the equilibrium allocation for any sequence of policy parameters (d_t, τ_t, e_t) . The welfare analysis considers variations in these parameters.

A2. Welfare Analysis

This appendix explains the welfare derivative (14). As explained in Section 4.2, Epstein-Zin (1989) type preferences are useful in the sensitivity analysis to calibrate the equity premium without linking risk-aversion to savings behavior (intertemporal substitution). To accommodate this generalization, the welfare analysis in this appendix is based on preferences of the form

$$U_t = \frac{1}{1-\beta} \left[(c_t^1) + \left\{ E_t[(c_{t+1}^2)^{1-\gamma}] \right\}^{1/(1-\gamma)} \right]^{1-\beta}$$

where β is the rate of time preference, γ is the degrees of risk aversion, and $1/(1-\gamma)$ is the elasticity of intertemporal substitution. These preferences yield the same allocation as (A.1a-i), except that all R_k terms must be replaced by $R_k/(1-\gamma)$.

In principle, the welfare function (13) could be maximized over a variety of policy instruments, either chosen period-by-period or fixed for all times. Here I consider a marginal variation in a single policy parameter, taking all others as given. This choice setting is most relevant for the main application, social security equity investments--assuming the social security administration does not control other policy instruments.³

² The scaling by lagged productivity avoids an awkward property of productivity ratios: Generally, a positive productivity shock a_t has negative effect on ratio variables like y_t^1 , c_t^1/A_t , c_t^2/A_t , and k_{t+1} (see, e.g., y_{1a} in (A.1e)) even though it raises the levels of income, consumption, and capital. In contrast, $c_{1a} = c_{1Aa}+1$ is positive for reasonable parameters.

³ An alternative would be to examine the simultaneous choice of all policy parameters, but that would implicitly assume considerable policy coordination and, with a sufficient set

I further focus a one-time change in the allocation of risk, to highlight that even one-time changes have long-lasting effects. Multi-period or permanent changes could always be interpreted as a succession of one-period changes. Specifically, I assume that at time $t=0$, the government changes some policy parameter so that the period-1 exposure of the old generation 0 and the young generation 1 to shocks is altered. (After period 1, the allocation of risk remains unchanged.) I consider a generic parameter to show that the approach is quite general and could, e.g., be used to examine a variety of debt and tax policies. The application to social security equity investments is obtained by setting $\theta = \theta_0 e_0/k$.

Since the focus of the paper is on risk-sharing and not redistribution, I assume that the overall scale of deterministic intergenerational redistribution matches the social planners welfare weights. That is, I focus on welfare weights such that the assumed size of social security (θ), the level of government debt (d) and the allocation of taxes (τ) would be efficient in a deterministic version of the model. Given the resource constraint

$$N_t c_t^1 + N_{t-1} c_t^2 + G_t + K_{t+1} = Y_t + v_t K_t,$$

deterministic efficiency implies the first order condition

$$(A.2) \quad (t) dU_t/dc_t^1 N_t/N_{t-1} = (t-1) dU_{t-1}/dc_{t-1}^2.$$

Since U_t is homogenous of degree $1-\alpha$, balanced growth and the transversality condition require that $(t) = \theta^t N_t$ is exponential and proportional to population and that $\theta^* = (1+n)(1+a)^{1-\alpha}$ satisfies $\theta^* \in (0,1)$. (Arbitrary welfare weights could be accommodated without changing essential results, if one took the perfect foresight path of the

of instruments, end up yielding the first-best allocation, which is more conveniently obtained directly; see Bohn (1998).

deterministic economy as baseline for the log-linearization. But that would complicate the exposition without providing new insights.)

A subtle point concerns government spending. Spending in proportion to GDP is reasonable for modeling the time series of government spending, but endogenous spending would distort the planner's problem just like a tax on output. Hence, I assume that spending is proportional to the GDP of the original allocation, but exogenous at that level and not varying with alternative policies.

In general, the welfare effect of a period-0 change in a policy parameter is given by differentiating (A.2),

$$(A.3) \quad \frac{dW_0}{d} = E_0 \left[\sum_{t=0}^{\infty} \beta^t N_t \frac{dU_t}{d} \right] = N_0 E_0 \left[\frac{dU_0}{dc^2_1} \frac{dc^2_1}{d} \right] \\ + E_0 \left[\sum_{t=1}^{\infty} \beta^t N_t \left\{ \frac{dU_t}{dc^1_t} \frac{dc^1_t}{d} + \frac{dU_t}{dc^2_{t+1}} \frac{dc^2_{t+1}}{d} \right\} \right]$$

subject to the macroeconomic dynamics approximately characterized by (A.1a-h). The period-1 policy change has direct effects on the state-contingent consumption at time $t=1$, i.e., on c^1_1 and c^2_1 only. But additional "indirect" effects arise changes in the period-1 income of the young affect the state-contingent path of capital accumulation; generally $dc^i_t/d = dc^i_t/dk_t dk_t/d$ is non-zero for all $t \geq 1$.

In the context of a first-best welfare maximization problem, the indirect effects could be ignored with reference to the envelope theorem. Future policy choices would be such that the benefits of capital accumulation are allocated efficiently across future generations. But if future policy is taken as given, changes in future capital stocks induced by a $t=1$ reallocation of risk will have a non-trivial impact on future generations.

The envelope theorem can be used, however, to write the impact on generations $t+1$ as

$$\begin{aligned}
 (A.4) \quad E_0 \left[\frac{dU_t}{d} \right] &= {}^t N_t E_0 \left[\left\{ \frac{dU_t}{dc^1_t} \frac{dc^1_t}{dk_t} + \frac{dU_t}{dc^2_{t+1}} \frac{dc^2_{t+1}}{dk_t} \right\} \frac{dk_t}{d} \right] \\
 &= E_0 \left[\left\{ \frac{dU_t}{dc^1_t} A_t \frac{dy^1_t}{dk_t} + \frac{dU_t}{dc^2_{t+1}} \left(\frac{dc^2_{t+1}}{dk_t} - R^k_{t+1} A_t \frac{dk_{t+1}}{dk_t} \right) \right\} \frac{dk_t}{d} \right] \\
 &= u^1_t - (1+n) u^2_{t+1},
 \end{aligned}$$

where $u^1_t = E_0 \left[\frac{dU_t}{dc^1_t} A_t \frac{dy^1_t}{dk_t} \frac{dk_t}{d} \right]$ and $u^2_{t+1} = E_0 \left[\frac{dU_t}{dc^2_{t+1}} A_{t+1} \frac{dy^1_{t+1}}{dk_{t+1}} \frac{dk_{t+1}}{d} \right]$.

The first equality holds because generation t optimizes over k_{t+1} and $dc^1_t/dk_t = A_t (dy^1_t/dk_t - dk_{t+1}/dk_t)$; the second follows from the resource and technology constraints.⁴ Thus, the indirect welfare effects depend on how changes in capital affect future generations' disposable income y^1 . If (A.4) is used in (A.3) and the sum is re-arranged by periods (rather than by cohort), one obtains

$$\begin{aligned}
 (A.3') \quad \frac{dW_0}{d} &= N_0 E_0 \left[\frac{dU_0}{dc^2_1} \frac{dc^2_1}{d} \right] + E_0 \left[\sum_{t=1} {}^t N_t \left\{ u^1_t - (1+n) u^2_{t+1} \right\} \right] \\
 &= N_0 \left(E_0 \left[\frac{dU_0}{dc^2_1} \frac{dc^2_1}{d} \right] + (1+n) u^1_1 \right) + \sum_{t=2} {}^{t-1} N_t (u^1_t - u^2_t).
 \end{aligned}$$

The terms $u^1_t - u^2_t$ would be identically zero in a first-best allocation, because a first-best allocation would require (A.2) to hold in every state of nature (see Bohn 1998), which implies $u^1_t = u^2_t$. But for given policy rules, (A.2) is non-zero along most sample paths--even though I assume that (A.2) is zero in expectation to remove pure redistributive issues. Given

⁴ Let CF_{t+1} be the cash-flows from the government to the old (all non-capital income), so that $c^2_{t+1} = R^k_{t+1} (k_{t+1} A_t) + CF_{t+1}$, then

$$\frac{dc^2_{t+1}}{dk_t} = [R^k_{t+1} A_t + k_{t+1} A_t \frac{dR^k_{t+1}}{dk_{t+1}} + \frac{dCF_{t+1}}{dk_{t+1}}] \frac{dk_{t+1}}{dk_t}$$

By Euler's law, $\frac{k_{t+1}}{1+at+1} \frac{dR^k_{t+1}}{dk_{t+1}} = -(1+n) \frac{d(w/A)_{t+1}}{dk_{t+1}}$, so that

$$\frac{dc^2_{t+1}}{dk_t} - R^k_{t+1} A_t \frac{dk_{t+1}}{dk_t} = - \left((1+n) A_{t+1} \frac{d(w/A)_{t+1}}{dk_{t+1}} - \frac{dCF_{t+1}}{dk_{t+1}} \right) \frac{dk_{t+1}}{dk_t}.$$

Since cash-flows to the old must come from the young (by the government budget identity, holding real spending constant), the term in bracket is

$$(1+n) A_{t+1} \frac{d(w/A)_{t+1}}{dk_{t+1}} - \frac{dCF_{t+1}}{dk_{t+1}} = (1+n) A_{t+1} \frac{dy^1_{t+1}}{dk_{t+1}}$$

which explains the format of u^2_{t+1} and its negative sign.

(A.2) is zero in expectation, the expressions in (A.3') can be interpreted as covariances between marginal utilities and policy-induced changes in income. Intuitively, a policy change "improves" the allocation of risk, if it gives additional income to a cohort in those states of nature in which its marginal utility is above the (\bar{u} -weighted) marginal utility of the other cohort.⁵

Similar arguments apply for the period-1 tradeoff between c^1_1 and c^2_1 , the first two terms in (A.3'). Since generation 1 optimizes over k_2 , the relevant tradeoff is between c^2_1 and y^1_1 . A policy change improves the allocation of risk in period 1, if it gives additional income to the old (raises c^2_1 and lowers y^1_1) in states of nature in which dU_0/dc^2_1 is above dU_1/dc^1_1 .

To obtain (14), consider the case of $\theta = e_0 \theta_0/k$ (so that θ can be interpreted as the fraction of the capital stock held by social security), use the macroeconomic dynamics of (A.1a-h), and take log-linear and log-normal approximations. Then (A.3') can be written as

$$(A.5) \quad \frac{dW_0}{d} = \bar{u} \frac{DW_a}{DW_\mu} COV_s \left(-\frac{d c_{2,s}}{d} \right)_s + \bar{u} DW_k \left(\frac{d k_{s}}{d} \right)_s COV_s \left(\frac{d y_{1k}}{d} \right)_s$$

where $\bar{u} = (dU_0/dc^2_1) c^2_1$ and $\theta = \frac{y^1_1}{(c^2/A)/(1+n)} \frac{y^1_1 k}{1 - \frac{y^1_1 k}{k k^2}}$ are constants; COV_s is the covariance matrix of the shocks $(\hat{a}_1, \hat{v}_1, \hat{\mu}_1)$; $(\frac{d z_{s}}{d})_s$ is a gradient vector with elements $d x_{s}/d$ ($s=a,v,\mu$); $(\frac{d k_{s}}{d})_s$ is a vector with elements k_{s} . Finally, the DW-terms are

$$(A.6) \quad DW_s = (c^2_s - c^1_s) + (1 - \theta) (\theta + -1) / R_k k_s$$

where $\theta = (c^1/A) / [(c^1/A) + (c^2/A) (1+a)] \in (0,1)$ and

⁵ Here one can see that setting (A.4) to zero in expectation is conceptually useful to distinguish risk-sharing and redistributive issues. If (A.4) were violated in expectation, certain policy changes may be desirable or undesirable merely because of their deterministic distributional effects, which would confound their risk-sharing effects.

$$(A.7) \quad DW_k = (c_{2k} - c_{1k}) + (\alpha - 1)/R_k [(1 - \alpha) k_{k1} + 1].$$

In essence, dW_0/d is a weighted average of the DW-terms. For each of the shocks $(s=a, v, \mu)$, DW_s measures the discrepancy between dU_0/dc_1^2 and dU_1/dc_1^1 in states of nature in which the respective shock is non-zero. For example, consider a productivity shock $s=a$. For CRRA utility, the DW-terms reduce to $DW_s = c_{2s} - c_{1s}$, so that (A.5) reduces to (14).

A3. Derivation of the Earnings-Income Ratio

Let earnings E_t^f be the capital income of the firms in the social security equity portfolio (Y_t^f) minus accounting depreciation (a constant DEP) minus interest expenses,

$$E_t^f = Y_t^f - DEP K_t^f - (R_t^b - 1)(-1)/K_t^f$$

where K_t^f are the firm's assets, which are α times the firm equity. The firm's capital income is the income component of (10). It equals the firm's capital stock times the aggregate capital income/capital stock ratio times the relative shock, $Y_t^f = Y_t/K_t K_t^f \mu_t$. Using a log-linear approximation around the steady state (around $Y^f/K^f = R^{k-1} + DEP$), one can write the ratio of firms earnings to aggregate capital income as

$$(A.8) \quad \ln\left(\frac{E_t^f}{Y_t}\right) = \ln\left(\frac{E_t^f}{K_t^f}\right) - \ln\left(\frac{Y_t}{K_t^f}\right) \\ \ln\left(\frac{K_t^f}{K_t}\right) + (\alpha - 1) \ln\left(\frac{Y_t}{K_t}\right) + \alpha \ln(\mu_t) - \frac{R^k}{R^{k-1}} \ln(R_t^b)$$

where $\alpha = \frac{R^{k-1} + DEP}{R^{k-1}} > 0$ is the steady state ratio of firm earnings to capital income. In terms of innovations, unexpected changes in the earnings-income ratio are a linear combination of productivity shocks and relative earnings shocks, namely

$$(A.9) \quad (\hat{E^f/Y})_{t-E_{t-1}}[(\hat{E^f/Y})_t] = (\alpha - 1)(1 - \alpha) \hat{a}_t + \alpha \hat{\mu}_t,$$

as claimed in the text (using $Y_{t-E_{t-1}}[Y_t] = (1 - \alpha) \hat{a}_t$). Note that the ratio of firm size K_t^f to the aggregate capital stock K_t is unrestricted in this

derivation. In the data, the ratio of S&P500 earnings to aggregate capital income displays a negative time trend. In the context of (A.8), this trend can be interpreted as a trend in the relative capital stocks, and hence, as consistent with the theoretical model.